

Geometry of Principal Fiber Bundles

Homogeneous Spaces, Invariant Connections, Gauge Theory,
and Beyond

DRAFT VERSION

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April 8, 2026

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